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ON SOLUTIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS $f'(x) = a(x)f(g(x)) + b(x)f(x) + c(x)$ IN THE LARGE

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ABSTRACT

We consider solutions of functional-differential equations

$$f'(x) = a(x)f(g(x)) + b(x)f(x) + c(x)$$

in both real and complex variables. We characterize entire solutions g when f is a meromorphic function in the complex plane and $a \neq 0, b, c$ are constants or polynomials. We also examine questions of existence and uniqueness of the solutions in the real variable for initial value problems and provide theorems that are valid "in the large".

1.

The purpose of this article is to study existence and uniqueness problems, and to characterize solutions of functional-differential equations

$$f'(x) = af(g(x)) \tag{1.1}$$

and their generalizations

$$f'(x) = af(g(x)) + bf(x) + c, \tag{1.2}$$

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where $a \neq 0, b, c$ are constants, or more generally, functions. Such equations appear in the theory of boundary value problems of hyperbolic partial differential equations and include the pantograph equations $f'(x) = af(\alpha x) + bf(x)$ (a, b, α are constants) as a special case, which have numerous applications ranging from cell growth models to current collection systems for an electric locomotive to wavelets (see e.g. [OT] and [W] and references therein) and have been studied for both real and complex variables by numerous authors. When $g(x) = x - k, k > 0$ a fixed number, the equation (1.1) is the well-known and extensively studied linear differential-difference (or delay-differential) equation (see [BC]). In [U], Utz posed the problem of existence for the equation (1.1). Siu ([S]) gave existence and uniqueness results for (1.1) that are global in nature under certain conditions on the function g and the constant a. The problems of local existence and uniqueness for more general equations were considered by Anderson in [A] and by Oberg in [O1] for local real solutions and in [O2] for local complex solutions. Related equations of complex variable were also considered in [B], [BMW], [D], [DI], [G], [GY], etc. It was proved in [G] that if f is a nonconstant entire function and g is an entire function in the complex plane $\mathbf{C}$ satisfying the equation (1.1) with a being a constant, then g must be linear. The same conclusion was extended recently in [BMW] to the equations $f' = af(g) + bf$ with $a \neq 0, b$ being constants, which thus reduce to pantograph type equations in this case. However, no such characterization is known when f is a meromorphic function in $\mathbf{C}$; it is only known ([GY]) that g is a polynomial when f is a transcendental meromorphic function in $\mathbf{C}$.

Despite the large number of studies concerning these equations, and the occasional overlap of some of the results, some questions still remain unsolved. In particular, it is unknown whether g is linear in (1.1) when f is a transcendental meromorphic function in $\mathbf{C}$. This is one of the problems we are going to solve in this paper. We present our results in two sections. In Section 2, we characterize entire solutions g for (1.1) and also (1.2) when f is a meromorphic function in $\mathbf{C}$; and in Section 3, we investigate the existence and uniqueness problems to such equations for $x \in [0, L] \subset \mathbf{R}$.

To establish our results, we need to employ different analytical tools and techniques in the following sections, including Nevanlinna theory in Section 2, and the operator theory in Section 3. We note, however, that the problem and the results in Section 2 are independent of Nevanlinna theory; it would be interesting to discover an elementary proof without using Nevanlinna theory.

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2.

In this section we will characterize entire functions g for the equations (1.1) and more general equations (1.2) when f is a meromorphic function in $\mathbf{C}$ and a, b, c are constants (Theorem 2.1) or, more generally, polynomials (Theorem 2.4).

When f is a nonconstant entire function in (1.1), it is known that only entire solutions g are linear functions ([G], [BMW]). This result is however false when f is a meromorphic function (see below). We will see that whether f is transcendental or not also makes the situations different.

For clarity, the independent variable will be denoted by z when it is complex.

THEOREM 2.1: *Suppose that f is a nonconstant meromorphic function and g is an entire function in $\mathbf{C}$ satisfying the equation $f'(x) = af(g(z)) + bf(z) + c$ with $a \neq 0, b, c$ being constants. Then*

(i) *g must be linear, if f is transcendental;*

(ii) *g must be a polynomial of degree less than or equal to 2, if f is rational; furthermore, the degree of g is 2 if and only if $f = \frac{\alpha}{z-w_0} + \beta$, $g = w_0 - a(z-w_0)^2$ and $b = a\beta + c = 0$, where $\alpha \neq 0, \beta, w_0$ are complex numbers.*

Remark: The equations in Theorem 2.1 include Eq. (1.1) as a special case. We see that g might be nonlinear when f is meromorphic. A result on the growth of f can be obtained from [GY] when g is a nonlinear entire function and f is a transcendental meromorphic function in $\mathbf{C}$. However, we now see from Theorem 2.1 (i) that when g is nonlinear, f cannot be transcendental. In fact, when g is nonlinear, Theorem 2.1(ii) completely characterizes f, which must be of the form $f = \frac{\alpha}{z-w_0} + \beta$ for some $\alpha \neq 0, \beta, w_0$.

From the above remark, we have the following

COROLLARY 2.2: *Suppose that f is a meromorphic function and g is an entire function in $\mathbf{C}$ satisfying the equation in Theorem 2.1. If g is nonlinear, then the solution f must be of the form $f = \frac{\alpha}{z-w_0} + \beta$ for some constants α, β, w_0.*

If $b \neq 0$, then by Theorem 2.1(ii), g cannot be nonlinear. Thus, we have the following

COROLLARY 2.3: *Let $b \neq 0$ in the equation in Theorem 2.1. Suppose that f is a meromorphic function and g is an entire function in* **C** *satisfying the equation. If g is nonlinear, then the solution f must be a constant.*

Remark: The condition $b \neq 0$ cannot be dropped in Corollary 2.3, as seen in Theorem 2.1 (ii). In fact, if $b = 0$, then $f = 1/z - c/a$, which is nonconstant, and $g = -az^2$, which is nonlinear, satisfy the equation $f' = af(g) + c$, which is of the form of the equations in Theorem 2.1 with $b = 0$.

To prove Theorem 2.1, we will employ Nevanlinna theory. For the reader's convenience, we recall some notation and results in Nevanlinna theory (see e.g. [Y]), which will be needed in the proof of Theorem 2.1. Let f be a meromorphic function in **C**. Then the Nevanlinna characteristic $T(r, f)$ is defined as

$$T(r,f) = m(r,f) + N(r,f),$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta; \quad \log^+ |x| = \max(0, \log |x|)$$

and

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,r)\log r;$$

$n(t, f)$ denotes the number of poles of f (counting multiplicity) in $|z| < r$. Recall the following known results:

(i) $T(r, f)$ is an increasing convex function of $\log r$. $T(r, f) = T(r, 1/f)+O(1)$. $T(r, fg) \leq T(r, f) + T(r, g), T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$. The last two inequalities also hold for $m(r, f)$. See [Y, pp. 8–9, p. 12] for these results.

(ii) f is transcendental if and only if ([Y, p. 25])

$$\underline{\lim}_{r\to\infty} \frac{T(r,f)}{\log r} = \infty. \tag{2.1}$$

(iii) The logarithmic derivative lemma([Y, p. 17]):

$$m(r, f'/f) = o\{T(r,f)\} \tag{2.2}$$

for all r outside possibly a set of finite Lebesgue measure.

(iv) If f (meromorphic) and g (entire) are transcendental, then

$$\lim_{r\notin E} \sup_{r\to\infty} \frac{T(r, f(g))}{T(r,f)} = \infty, \tag{2.3}$$

where E is any set of finite Lebesgue measure (see [C, Theorem 2] and [GY, p. 370]).

(v) If $g = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ is a nonconstant polynomial, then for any $\epsilon > 0$,

$$T(r, f(g)) \geq (1-\epsilon)T\Big(\frac{a_m}{2}r^m, f\Big) \tag{2.4}$$

for large r (see [GY, (19)]).

Proof of Theorem 2.1: First of all, we have that

$$\begin{aligned} T(r, f'-bf) &= N(r, f'-bf) + m(r, f'-bf) \\ &\leq 2N(r,f) + m\Big(r, \frac{f'-bf}{f}\Big) + m(r,f) \\ &\leq 2T(r,f) + o\{T(r,f)\} \end{aligned}$$

outside possibly a set of finite Lebesgue measure, by (2.2). We then have that

$$\begin{aligned} T(r, f(g)) = T\Big(r, \frac{f'-bf}{a} - \frac{c}{a}\Big) &\leq T(r, f'-bf) + O(1) \\ &\leq 2T(r,f) + o\{T(r,f)\} \end{aligned} \tag{2.5}$$

outside possibly a set of finite Lebesgue measure. We then see that g must be a polynomial, sine if g was transcendental, then it is easy to see from the given equation $f'(x) - bf - c = af(g(z))$ that f would be also transcendental, which is impossible by (2.5) and (2.3).

Now, we write $g = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$. If $m \leq 1$, then the conclusions (i) and (ii) of Theorem 2.1 both already hold. Thus, in the following we assume that $m \geq 2$. We will show that $m = 2$ and f cannot be transcendental. This, of course, will imply (i) and the first part of (ii) of the theorem.

By (2.4), we have that

$$T(r, f(g)) \geq (1-\epsilon_1)T(a_m/2r^m, f) \tag{2.6}$$

for any $\epsilon_1 > 0$. Recall that $T(r,f)$ is a convex function of $\log r$. Thus, for large r, we have that

$$\frac{T(r,f) - T(1,f)}{\log r - \log 1} \leq \frac{T(a_m/2r^m, f) - T(1,f)}{\log(a_m/2r^m) - \log 1},$$

which implies that

$$T(r,f) \leq \frac{1}{m}(1+\epsilon_2)T(a_m/2r^m, f) \tag{2.7}$$

for any $\epsilon_2 > 0$ and large r. Combining this with (2.5) and (2.6) we obtain that

$$mT(r,f) \leq \frac{1+\epsilon_2}{1-\epsilon_1}(2T(r,f) + o\{T(r,f)\})$$

for large r outside possibly a set of finite Lebesgue measure. This implies that $m \leq 2$.

From above, we have proved that $m = 2$. We write g as $g(z) = \alpha(z-z_0)^2 + \beta$ with $\alpha \neq 0, \beta, z_0$ being complex numbers.

We claim that f has at most one pole. Suppose that f has at least two distinct poles. We can arrange all the distinct poles of f according to their orders in an increasing order: $a_1, a_2, \ldots, a_n, \ldots$ with orders $m_1, m_2, m_3, \ldots$, respectively, where $m_1 \leq m_2 \leq m_3 \leq \cdots$. We then write $f = \frac{1}{(z-a_1)^{m_1}(z-a_2)^{m_2}} A(z)$, where $A(z)$ is a meromorphic function, which is analytic at a_1, a_2 and $A(a_1) \neq 0$, $A(a_2) \neq 0$. Then, the given equation $f' = af(g) + bf + c$, or $f' - bf = af(g) + c$, becomes

$$\frac{B(z)}{(z-a_1)^{m_1+1}(z-a_2)^{m_2+1}} - \frac{bA(z)}{(z-a_1)^{m_1}(z-a_2)^{m_2}} = \frac{aA(g(z))}{(g(z)-a_1)^{m_1}(g(z)-a_2)^{m_2}} + c \tag{2.8}$$

where $B(z)$ is a meromorphic function, which is analytic at a_1, a_2 and satisfies that $B(a_1) \neq 0, B(a_2) \neq 0$. Now consider two cases: $\beta \neq a_1$ and $\beta = a_1$, where β is the number in the above expression of g. If $\beta \neq a_1$, then $g(z) - a_1 = 0$ has two distinct roots, which are both poles of the right hand side of (2.8) with order m_1. But, all the poles of the left hand side of (2.8) have order at least $m_1 + 1$, a contradiction. If $\beta = a_1$, then $\beta \neq a_2$. Thus, $g(z) - a_2 = 0$ has two distinct zeros, which, for the same reason as above, are both poles of the right hand side of (2.8) with order m_2. But, all the poles of the right hand side of (2.8) have order at least $m_2 + 1 > m_2$, except the pole a_1 with order $m_1 + 1$, which is possibly less than $m_2 + 1$. That is, the left hand side of (2.8) cannot have two distinct poles with order m_2. This is impossible. We have thus showed that f has at most one pole. Therefore, we can write $f(z) = Q(z)h(z)$, where $Q = \frac{1}{(z-w_0)^l}$ for some positive integer l, and h is an entire function with $h(w_0) \neq 0$.

Suppose now that f is transcendental. Then h is transcendental. The original equation in the theorem can be written as

$$Q'h + Qh' = aQ(g)h(g) + bQh + c,$$

or

$$h(g) = \frac{Q'h + Qh' - bQh - c}{aQ(g)} = h\Big(\frac{Q' + Q\frac{h'}{h} - bQ}{aQ(g)}\Big) - \frac{c}{aQ(g)}, \tag{2.9}$$

from which it follows that

$$\begin{aligned} T(r, h(g)) = m(r, h(g)) &\le m(r, h) + m(r, \frac{h'}{h}) + O\{\log r\} \\ &\le T(r, h) + o\{T(r, h)\}, \end{aligned}$$

for all r outside a set of finite Lebesgue measure by (2.1) and (2.2). But, by (2.4), we have that

$$T(r, h(g)) \ge (1 - \epsilon_3)T(\frac{\alpha}{2}r^2, h)$$

for any $\epsilon_3 > 0$ and large r. Also, by the convexity of $T(r, h)$ in $\log r$, we have that

$$T(r, h) \le \frac{1}{2}(1 + \epsilon_4)T(\frac{\alpha}{2}r^2, h)$$

for any $\epsilon_4 > 0$ and large r (cf. the proof of (2.7)). These inequalities yield that

$$T(r, h) \le \frac{1}{2}\frac{1 + \epsilon_4}{1 - \epsilon_3}(T(r, h) + o\{T(r, h)\}).$$

This implies that $T(r, h) = o\{T(r, h)\}$, which is absurd. This proves that f cannot be transcendental and thus proves the conclusion (i) of the theorem.

Next, if $f = Qh = \frac{1}{(z-w_0)^l}h$, defined above, is a rational function and g is a polynomial of degree 2, then h is a polynomial. By comparing the order of the pole w_0 in the two sides of the given equation $f' - bf = af(g) + c$ and noting that g has degree 2, we must have that $l + 1 = 2l$, i.e., $l = 1$. Then by the given equation again, we have that

$$\frac{(z - w_0)h' - h}{(z - w_0)^2} = a\frac{h(g)}{g - w_0} + b\frac{h}{z - w_0} + c$$

or

$$\begin{aligned} &((z - w_0)h' - h)(g - w_0) = \\ &\qquad ah(g)(z - w_0)^2 + bh(z - w_0)(g - w_0) + c(z - w_0)^2(g - w_0). \end{aligned} \tag{2.10}$$

Suppose that the degree of h is $d \ge 0$. If $d \ge 2$, then it is easy to check that the left hand side of (2.10) has degree $d+2$, while the right hand side has degree $2d + 2$, which is impossible. Thus, we have that $d = 0$ or $d = 1$.

We can then write $f=\beta+\frac{\alpha}{z-w_0}$ for some constants $\alpha\neq 0,\beta$. The given equation can be then written as

$$-\frac{\alpha}{(z-w_0)^2}=\frac{a\alpha}{g-w_0}+\frac{b\alpha}{z-w_0}+c+(a+b)\beta,$$

from which it follows that

$$g-w_0=\frac{a\alpha(z-w_0)^2}{-\alpha-b\alpha(z-w_0)-((a+b)\beta+c)(z-w_0)^2}.$$

But, g is a polynomial of degree 2. Thus, we must have that $b\alpha=(a+b)\beta+c=0$, which implies that $b=0$, $a\beta+c=0$, and then $g=w_0-a(z-w_0)^2$. Therefore, f and g are of the forms in Theorem 2.1 (ii).

Conversely, if f and g have the forms in Theorem 2.1 (ii), it is easy to check that f and g satisfy the given equation. This proves Theorem 2.1 (ii). The proof of the theorem is thus complete. ∎

Using the method in the proof of Theorem 2, we may treat similar types of equations with f' replaced by a polynomial or a rational function of $z,f,f',\ldots$, $f^{(k)}$ with coefficients being polynomials or appropriate meromorphic functions growing more slowly than f.

We include an extension of Theorem 2.1 to the equations (1.2) with polynomial coefficients.

THEOREM 2.4: *Suppose that f is a transcendental meromorphic function in $\mathbf{C}$ and g is an entire function satisfying the equation*

$$f'(z)=a(z)f(g(z))+b(z)f(z)+c(z),$$

where $a\not\equiv 0,b,c$ are polynomials in $\mathbf{C}$. Then g must be linear.

Remark: (i) Unlike Theorem 2.1, when f is rational in Theorem 2.4, g may be a polynomial of any given degree. For example, for any given integer $n\geq 1$, $f=1/z$ and $g(z)=z^n$ satisfy the equation $f'=\lambda f(g)$ with $\lambda=-z^{n-2}$.

(ii) Theorem 2.4 is false if without certain restriction on the growth of the coefficients a,b,c. For example, $f(z)=e^z+1$, and $g(z)=z\cos^2 z$, which are nonlinear, satisfy the equation $f'=\lambda f(g)+k$ with $\lambda(z)=-k(z)=e^{z\sin^2 z}$.

Proof of Theorem 2.4: The proof is similar to the one of Theorem 2.1, using the fact that $T(r,p)=O\{\log r\}=o\{T(r,f)\}$ for any polynomial p. This fact implies that (2.5) still holds. We also have (2.6) and (2.7), as in the proof of Theorem 2.1. We then obtain that g is a polynomial of degree $m\leq 2$. If

$m = 2$, following the same arguments in Theorem 2.1, we deduce that $T(r,h) = o\{T(r,h)\}$, where h is a transcendental entire function, as defined in Theorem 2.1. This is impossible. Therefore, $m = 1$, i.e., g is linear. ∎

3.

In this section we will provide existence and uniqueness results that hold "in the large" on the whole interval $[0,L] \subset R$ ($L > 0$ a constant). We first show global existence (see Theorem 3.1 (i)) for continuous functions g satisfying that $g([0,L]) \subset [0,L]$, which needs to be assumed so that the both sides of the equation $f'(x) = a(x)f(g(x))+b(x)f(x)+c(x)$ make sense on the interval $[0,L]$. Then we consider the question of uniqueness for the global solutions (Theorem 3.2(ii)), which seems to be more subtle. We will consider the uniqueness question for the equations $f'(x) = a(x)f(g(x)) + b(x)$. In part (a) of Theorem 3.1(ii), g is assumed to satisfy $g(x) \leq x$; and in part (b), g takes the form of $x^\alpha, \alpha > 0$, which may satisfy that $g(x) \geq x$. When $g(x) \geq x$, the equation is an functional differential equation with an advanced argument. The proof of the uniqueness presented here is elementary. The method may also work in more general cases for which the calculation can go through. Theorem 3.1 should be compared to the results of Siu ([S]), where the equation $f'(x) = af(g(x))$ with a constant a was considered and the results depend on the size of the coefficient a, and to the results of Oberg ([O1]), where more general equations were considered but only for local solutions.

Let $C([0,L])$ be the space of real-valued continuous functions on $[0,L]$, and $C^1([0,L])$ the space of real-valued functions with continuous derivative on $[0,L]$. It is well-known that with the supremum norm, $||f|| := \sup_{0\leq x\leq L}|f(x))|$, $C([0,L])$ is a Banach space.

THEOREM 3.1: *Suppose that the functions a,b,c,g belong to $C([0,L])$, and $g([0,L]) \subseteq [0,L]$. Then*

(i) *the equation $f'(x) = a(x)f(g(x)) + b(x)f(x) + c(x)$ has a solution f in $C^1([0,L])$ satisfying the initial condition $f(0) = f_0$ for any given constant f_0;*

(ii) *the equation $f'(x) = a(x)f(g(x)) + b(x)$ has at most one solution f in $C^1([0,L])$ satisfying the initial condition $f(0) = f_0$, if one of the following holds:*

  (a) *$g \in C^1([0,L])$ satisfies that $g(x) \leq x$ in $[0,L]$;*

  (b) *$g(x) = x^\alpha (\alpha > 0)$, provided that $L \leq 1$ when $\alpha > 1$, and $l < 1+\alpha$ when $\alpha < 1$, where $l := \max_{0\leq x\leq L}\{|a(x)|\}$.*

Proof: (i) We first prove Theorem 3.1 (i). Write the given equation as the integral equation

$$f(x) = f_0 + \int_0^x (a(s)f(g(s)) + b(s)f(s))ds + \int_0^x c(s)ds.$$

Define the operator S on functions h in $C([0, L])$ by

$$S(h)(x) = f_0 + \int_0^x (a(s)h(g(s)) + b(s)h(s))ds + \int_0^x c(s)ds,$$

for $x \in [0, L]$. Let B be a bounded subset of $C([0, L])$. For any $h \in B$, denote $f = S(h)$. We have that $||f'|| \leq M$, M a constant. By the Mean Value Theorem $|f(x) - f(x_1)| \leq M|x - x_1|$, for all x, x_1 in $[0, L]$. By the Ascoli–Arzèla theorem, for any bounded sequence $\{h\}$ in B, the sequence $\{S(h)\}$ has a convergent subsequence, and thus S is a compact operator from $C([0, L])$ to $C([0, L])$ (see e.g. [Be, p. 89 and p. 32]).

Now let $G = \{h \in C([0, L]) : |h(x)| \leq M_1 e^{2l_1 x}, x \in [0, L]\}$, where $M_1 = 2(KL + |f_0|)$, $K = \max_{0 \leq x \leq L} |c(x)|$, $l_1 = \max_{0 \leq x \leq L}\{|a(x)| + |b(x)|\}$. Then G is a closed bounded convex set in $C([0, L])$. For any h in G, we have

$$|f'(x)| \leq |a(x)h(g)(x)| + |b(x)h(x)| + |c(x)| \leq l_1 M_1 e^{2l_1 x} + K.$$

Hence,

$$\begin{aligned} |f(x)| &\leq \int_0^x |f'(s)|ds + |f_0| \leq \frac{l_1 M_1}{2l_1}(e^{2l_1 x} - 1) + KL + |f_0| \\ &\leq (KL + |f_0|)e^{2l_1 x} \leq M_1 e^{2l_1 x}, \quad \text{for } x \in [0, L]. \end{aligned}$$

Therefore, S maps G into itself.

By the Schauder Fixed Point Theorem (see e.g. [Be, p. 90]) that a compact map from a nonempty closed bounded convex subset of a Banach space to itself has a fixed point, we obtain a fixed point f of S. This function $f \in C^1([0, L])$ is a solution to the above integral equation and thus a solution of the given equation in the theorem satisfying the initial condition. This completes the proof of Theorem 3.1 (i).

(ii) Next we prove Theorem 3.1 (ii). First we prove part (a). Let $u, v \in C^1([0, L])$ be two solutions of the given equation satisfying the initial condition. Then for $0 \leq x \leq L$,

$$(3.1)\quad w(x) := u(x) - v(x) = \int_0^x a(s)[u(g(s)) - v(g(s))]ds = \int_0^x a(s)w(g(s))ds.$$

By the inequality: $2ab \le (a^2+b^2)$ and the Hörlder inequality, we obtain that

$$2w'(x)w(x) \le (w'(x))^2 + Ll^2 \int_0^x w^2(g(s))ds. \tag{3.2}$$

Let $W(x) = \int_0^x w^2(g(s))ds$. Then by (3.1), $(w'(x))^2 \le l^2 w^2(g(x)) = l^2 W'(x)$, and then by (3.2),

$$\begin{aligned} W''(x) &= 2w(g(x))w'(g(x))g'(x) \le l^2 W'(g(x))g'(x) + Ll^2 W(g(x))g'(x) \\ &\le l^2[W(g(x))]' + L_1 l^2 W(g(x)), \end{aligned} \tag{3.3}$$

where $L_1 = L \max_{x\in[0,L]}\{|g'(x)|\}$.

Let $p = -l\sqrt{l^2+4L_1}, q = \frac{l^2-p}{2}$. Then $p+2q = l^2$, which implies that $p+q = l^2 - q = \frac{p+l^2}{2}$ and $(p+q)q = \frac{(l^4-p^2)}{4} = -L_1 l^2$. Therefore, we obtain from (3.3) that

$$e^{-(p+q)x}\{-(p+q)([W(g(x))]' - qW(g(x)))\} + e^{-(p+q)x}\{W''(x) - q[W(g(x))]'\} \le 0,$$

from which it follows that

$$\{e^{-(p+q)x}([W(g(x))]' - qW(g(x)))\}' \le e^{-(p+q)x}\{[W(g(x))]'' - W''(x)\}$$

or

$$\{e^{-px}(W(g(x))e^{-qx})'\}' \le e^{-(p+q)x}\{W(g(x)) - W(x)\}''. \tag{3.4}$$

Note that $0 \le g(x) \le x$. Thus, $g(0) = 0$. Also, $W(0) = w(0) = 0$. We then use integration by parts in the both sides of (3.4) to deduce that

$$\begin{aligned} &e^{-px}(W(g(x))e^{-qx})' \\ &\qquad \le e^{-(p+q)}[W(g(x)) - W(x)]' + (p+q)[W(g(x)) - W(x)]e^{-(p+q)x} \\ &\qquad\quad + (p+q)^2 \int_0^x [W(g(s)) - W(s)]e^{-(p+q)s}ds \\ &\qquad \le e^{-(p+q)}[W(g(x)) - W(x)]' + (p+q)e^{-(p+q)x}[W(g(x)) - W(x)]. \end{aligned}$$

The last inequality uses the fact that

$$W(g(x)) - W(x) = \int_x^{g(x)} w^2(g(s))ds \le 0$$

since $g(x) \le x$. We then have that

$$(W(g(x))e^{-qx})' \le e^{-qx}[W(g(x)) - W(x)]' + (p+q)e^{-qx}[W(g(x)) - W(x)].$$

Integrating the both sides of this inequality over $[0,x]$, we obtain that

$$\begin{aligned} W(g(x))e^{-qx} \le & e^{-qx}[W(g(x)) - W(x)] \\ & + (p+2q)\int_0^x e^{-qx}[W(g(s)) - W(s)]ds \\ \le & 0, \end{aligned}$$

in view of the fact that $p + 2q = l^2 \ge 0$. We thus have that $W(g(x)) \le 0$ for all $x \in [0, L]$. Hence, $W(g(x)) \equiv 0$, which implies by (3.3) that $W''(x) \le 0$. Thus, $W'(x)$ is decreasing and then $W'(x) \le W'(0) = 0$, which implies that $W(x) \le W(0) = 0$. By the definition of $W(x)$, we obtain that $w(g(x)) = 0$ for all x, which implies that $w(x) = 0$ for all $x \in [0, L]$ by (3.1), i.e, $u(x) = v(x)$ for all $x \in [0, L]$. This completes the proof of Part (a).

Next, we prove part (b). If $\alpha = 1$, or $\alpha > 1$ and $L \le 1$, then it is clear that g satisfies the conditions of part (a). Thus, the uniqueness follows from part (a). Now we assume that $\alpha < 1$. Let u, v and w be as above. Then, we obtain, from (3.1), that

$$|w(x)| := |u(x) - v(x)| \le l \int_0^x |w(g(s))|ds. \tag{3.5}$$

From this inequality, we obtain that

$$|w(s^\alpha)| \le l \int_0^{s^\alpha} |w(s_1^\alpha)|ds_1. \tag{3.6}$$

Then, by (3.5) and (3.6), we deduce that

$$|w(x)| \le l^2 \int_0^x \int_0^{s_1^\alpha} |w(s_2^\alpha)|ds_2 ds_1.$$

Repeat this iteration process n times, we get

$$|w(x)| \le l^n \int_0^x \int_0^{s_1^\alpha} \cdots \int_0^{s_{n-2}^\alpha} \int_0^{s_{n-1}^\alpha} |w(s_n^\alpha)|ds_n \cdots ds_1.$$

As $|w(g(x))|$ is bounded on $[0, L]$ by a constant $M > 0$, it is not difficult to show from the above inequality that

$$\begin{aligned} |w(x)| \le [Ml^n x^{\alpha^{n-1}+\alpha^{n-2}+\cdots+\alpha^2+\alpha+1}] \div [(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1) \\ (\alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1) \cdots (\alpha^2 + \alpha + 1)(\alpha + 1)]. \end{aligned}$$

Since $\alpha < 1$, we deduce from the above inequality that

$$|w(x)| \le Ml^n \frac{x^{\frac{\alpha^n - 1}{\alpha - 1}}}{(\alpha+1)^{n-1}} = M(\alpha+1)\Big(\frac{l}{\alpha+1}\Big)^n x^{\frac{\alpha^n-1}{\alpha-1}}$$

As $\alpha < 1$ and $l < \alpha + 1$, $|w(x)|$ goes to zero as $n \to \infty$ for any fixed x. We thus obtain that $w(x) = 0$, i.e., $u(x) = v(x)$ for all $x \in [0, L]$. This completes the proof of Part (b). The proof of the theorem is thus complete. ∎

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